MAB241 COMPLEX VARIABLES

LAURENT SERIES

1 What is a Laurent series?

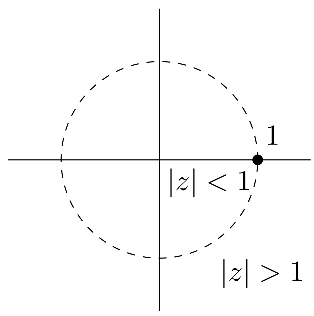
The Laurent series is a representation of a complex function $f(z)$ as a series. Unlike the Taylor series which expresses $f(z)$ as a series of terms with non-negative powers of z, a Laurent series includes terms with negative powers. A consequence of this is that a Laurent series may be used in cases where a Taylor expansion is not possible.

2 Calculating the Laurent series expansion

To calculate the Laurent series we use the standard and modified geometric series which are

$$\frac{1}{1-z} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} z^n, & |z| < 1, \\ \displaystyle -\sum_{n=1}^{\infty} \frac{1}{z^n}, & |z| > 1. \end{cases} \tag{1}$$

Here $f(z) = \frac{1}{1-z}$ is analytic everywhere apart from the singularity at $z = 1$. Above are the expansions for f in the regions inside and outside the circle of radius 1, centred on $z = 0$, where $|z| < 1$ is the region inside the circle and $|z| > 1$ is the region outside the circle.



2.1 Example

Determine the Laurent series for

$$f(z) = \frac{1}{(z+5)} \tag{2}$$

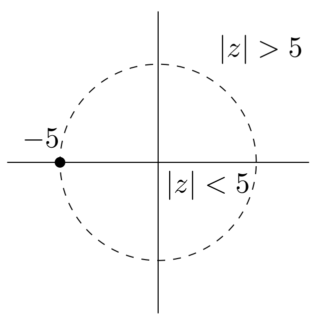
that are valid in the regions

(i) $\{z : |z| < 5\}$, and (ii) $\{z : |z| > 5\}$.

Solution

The region (i) is an open disk inside a circle of radius 5, centred on $z = 0$, and the region (ii) is an open annulus outside a circle of radius 5, centred on $z = 0$. To make the series expansion easier to calculate we can manipulate our $f(z)$ into a form similar to the series expansion shown in equation (1). So

$$f(z) = \frac{1}{5(1+\frac{z}{5})} = \frac{1}{5(1-(-\frac{z}{5}))}.$$



Now using the standard and modified geometric series, equation (1), we can calculate that

$$f(z) = \frac{1}{5(1-(-\frac{z}{5}))} = \begin{cases} \dfrac{1}{5}\displaystyle\sum_{n=0}^{\infty}\left(-\frac{z}{5}\right)^n, & |z| < 5, \\ -\dfrac{1}{5}\displaystyle\sum_{n=1}^{\infty}\frac{1}{\left(-\frac{z}{5}\right)^n}, & |z| > 5. \end{cases}$$

Hence, for part (i) the series expansion is

$$f(z) = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{z}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{5^{n+1}}, \qquad |z| < 5,$$

which is a Taylor series. And for part (ii) the series expansion is

$$f(z) = -\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{\left(-\frac{z}{5}\right)^n} = -\frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^n 5^n}{z^n} = -\sum_{n=1}^{\infty}\frac{(-1)^n 5^{n-1}}{z^n}, \qquad |z| > 5.$$

2.2 Example

Determine the Laurent series for

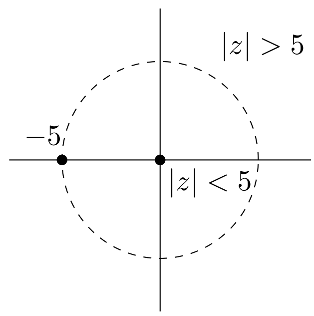
$$f(z) = \frac{1}{z(z+5)} \tag{3}$$

valid in the region $\{z : |z| < 5\}$.

<u>Solution</u>

We know from example 2.1 that for

$$\frac{1}{(z+5)}, \text{ the series expansion is } \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{5^{n+1}}, \qquad |z| < 5.$$



It follows from this that we can calculate the series expansion of $f(z)$ as

$$f(z) = \frac{1}{z}\cdot\frac{1}{(z+5)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{n-1}}{5^{n+1}}.$$

2.3 Example

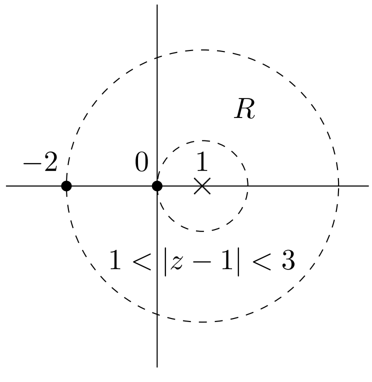
For the following function f determine the Laurent series that is valid within the stated region R.

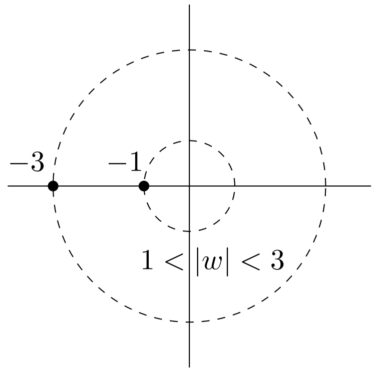
$$f(z) = \frac{1}{z(z+2)}, \ R = \{z : 1 < |z-1| < 3\}. \tag{4}$$

<u>Solution</u>

The region R is an open annulus between circles of radius 1 and 3, centred on $z = 1$. We want a series expansion about $z = 1$; to do this we make a substitution $w = z-1$ and look for the expansion in w where $1 < |w| < 3$. In terms of w

$$f(z) = \frac{1}{(w+1)(w+3)}.$$





To make the series expansion easier to calculate we can manipulate our $f(z)$ into a form similar to the series expansion shown in equation (1). To do this we will split the function using partial fractions, and then manipulate each of the fractions into a form based on equation (1), so we get

$$f(z) = \frac{1}{2}\left(\frac{1}{w+1} - \frac{1}{w+3}\right) = \frac{1}{2}\left(\frac{1}{1-(-w)} - \frac{1}{3(1-(-\frac{w}{3}))}\right).$$

Using the the standard and modified geometric series, equation (1), we can calculate that

$$\frac{1}{1-(-w)} = \begin{cases} \displaystyle\sum_{n=0}^{\infty}(-w)^n = \sum_{n=0}^{\infty}(-1)^n\, w^n, & |w| < 1, \\ \displaystyle -\sum_{n=1}^{\infty}\frac{1}{(-w)^n} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{w^n}, & |w| > 1, \end{cases}$$

and

$$\frac{1}{3(1-(-\frac{w}{3}))} = \begin{cases} \displaystyle\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{w}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n\, w^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\, w^n}{3^{n+1}}, & |w| < 3, \\ \displaystyle -\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{(-\frac{w}{3})^n} = -\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-3)^n}{w^n} = -\sum_{n=1}^{\infty}\frac{(-1)^n\, 3^{n-1}}{w^n}, & |w| > 3. \end{cases}$$

We require the expansion in w where $1 < |w| < 3$, so we use the expansions for $|w| > 1$ and $|w| < 3$, which we can substitute back into our $f(z)$ in partial fraction form to get

$$f(z) = \frac{1}{2}\left[-\sum_{n=1}^{\infty}\frac{(-1)^n}{w^n} - \sum_{n=0}^{\infty}\frac{(-1)^n\, w^n}{3^{n+1}}\right] = -\frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{w^n} + \sum_{n=0}^{\infty}\frac{(-1)^n\, w^n}{3^{n+1}}\right].$$

Substituting back in $w = z - 1$ we get the Laurent series, valid within the region $1 < |z-1| < 3$,

$$f(z) = -\frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{(z-1)^n} + \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}}\right].$$

2.4 Example

Obtain the series expansion for

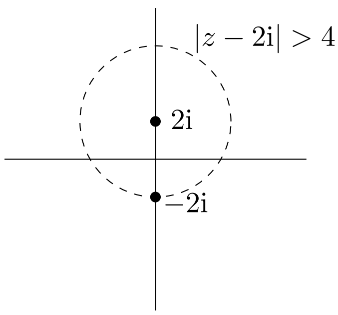
$$f(z) = \frac{1}{z^2+4} \tag{5}$$

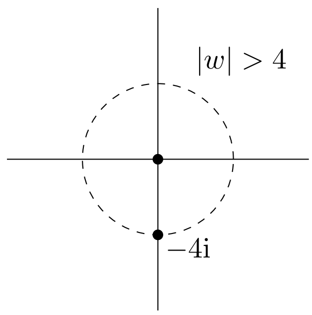
valid in the region $|z - 2\mathrm{i}| > 4$.

<u>Solution</u>

The region here is the open region outside a circle of radius 4, centred on $z = 2\mathrm{i}$. We want a series expansion about $z = 2\mathrm{i}$, to do this we make a substitution $w = z - 2\mathrm{i}$ and look for the expansion in w where $|w| > 4$. In terms of w

$$f(z) = \frac{1}{z^2+4} = \frac{1}{(z-2\mathrm{i})(z+2\mathrm{i})} = \frac{1}{w(w+4\mathrm{i})}.$$





To make the series expansion easier to calculate we can manipulate our $f(z)$ into a form similar to the series expansion shown in equation (1). To do this we will manipulate the fraction into a form based on equation (1). We get

$$f(z) = \frac{1}{4\mathrm{i}w\left(1 - (\frac{-w}{4\mathrm{i}})\right)} = \frac{1}{4\mathrm{i}w(1 - \frac{\mathrm{i}w}{4})}.$$

Using the the standard and modified geometric series, equation (1), we can calculate that

$$\frac{1}{4\mathrm{i}w(1 - \frac{\mathrm{i}w}{4})} = \begin{cases} \frac{1}{4\mathrm{i}w}\sum_{n=0}^{\infty}\left(\frac{\mathrm{i}w}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(\mathrm{i}w)^{n-1}}{4^{n+1}}, & |w| < 4, \\ -\frac{1}{4\mathrm{i}w}\sum_{n=1}^{\infty}\frac{1}{(\frac{\mathrm{i}w}{4})^n} = -\frac{1}{4\mathrm{i}w}\sum_{n=1}^{\infty}\left(\frac{4}{\mathrm{i}w}\right)^n = -\sum_{n=1}^{\infty}\frac{4^{n-1}}{(\mathrm{i}w)^{n+1}}, & |w| > 4. \end{cases}$$

We require the expansion in w where $|w| > 4$, so

$$f(z) = -\sum_{n=1}^{\infty}\frac{4^{n-1}}{(\mathrm{i}w)^{n+1}} = -\sum_{n=2}^{\infty}\frac{4^{n-2}}{(\mathrm{i}w)^n}.$$

Substituting back in $w = z - 2\mathrm{i}$ we get the Laurent series valid within the region $|z - 2\mathrm{i}| > 4$

$$f(z) = -\sum_{n=2}^{\infty}\frac{4^{n-2}}{(\mathrm{i}(z - 2\mathrm{i}))^n}.$$

3 Key points

- First check to see if you need to make a substitution for the region you are working with, a substitution is useful if the region is not centred on $z = 0$.
- Then you will need to manipulate the function into a form where you can use the series expansions shown in example (1): this may involve splitting by partial fractions first.
- Find the series expansions for each of the fractions you have in your function within the specified region, then substitute these back into your function.
- Finally, simplify the function and, if you made a substitution, change it back into the original variable.

For more information on Laurent series refer to the lecture notes.